Fractal traps and fractional dynamics

Pierre Inizan

FRACTAL TRAPS AND FRACTIONAL DYNAMICS

PIERRE INIZAN

Abstract. Anomalous diffusion may arise in typical chaotic Hamiltonian systems. According to G.M. Zaslavsky's analysis, a description can be done by means of fractional kinetics equations. However, the dynamical origin of those fractional derivatives is still unclear. In this talk we study a general Hamiltonian dynamics restricted to a subset of the phase space. Starting from R. Hilfer's work, an expression for the average infinitesimal evolution of trajectories sets is given by using Poincaré recurrence times. The fractal traps within the phase space which are described by G.M. Zaslavsky are then taken into account, and it is shown that in this case, the derivative associated to this evolution may become fractional, with order equal to the transport exponent of the diffusion process.

1. Introduction

Fractional calculus [26, 21, 22] is efficiently used in several fields of physics [12, 24]. For example, it may be used to take into account memory effects and anomalous transport. Several equations of physics have hence been generalized to non-integer orders so as to provide new models. Among them, one find the Euler-Lagrange equation [23, 1, 2, 5, 7] and the diffusion equation [20, 4, 9, 18, 27, 29, 6].

However, reasons for emergence of such operators are still unclear and the use this formalism often remains heuristic. R. Hilfer [10, 14, 11] and G.M. Zaslavsky [31, 34, 25, 33] have tried through different ways to understand the physical origin of fractional derivatives. Both of their models rely on the *recurrence time* notion.

The first of those authors studies the evolution operator of a subsystem and shows that after a temporal renormalization, the associated infinitesimal generator is a fractional derivative. However, the interpretation of this operator may seem difficult and the renormalization procedure is ambivalent.

Zaslavsky is interested in chaotic hamiltonian systems. He makes fractional derivatives appear in the diffusion equation related to the kinetic (i.e. probabilistic) description of the system. Without a true justification for the introduction of those derivatives, he nevertheless connects the transport exponent μ with the fractional orders of derivation and the coefficients of the self-similar structures which appear in the phase space around resonance areas.

In the present article, we study the dynamics of an Hamiltonian system, presented in section 2. With ideas taken from Hilfer, we focus in section 3 on the evolution of a phase space subset under the hamiltonian flow. More precisely, the associated infinitesimal generator is considered. In several examples we show that it is proportional to the usual derivative d/dt. Then we precise our model by taking into account the phase space structure described by Zaslavsky and sumed up in section 4. In that case, we prove in section 5 that the infinitesimal generator may turn into a fractional derivative of order μ. We discuss the relevance of this exponent in section 6, before concluding in section 7.

2. Studied system

Let H be an Hamiltonian defined on a compact manifold M. The induced flow is denoted ϕ^t. Let m be a measure defined on M. Let G be a measurable subset of M and m' a measure adapted to G (such that $m'(G) > 0$). Two cases may happen: if $m(G) > 0$, then we can choose $m' = m$. Conversely if $m(G) = 0$ (important case in this article), m cannot measure subsets of G, so m' must differ from m.

Let us suppose that we only have access to G. Thus we are interested in the dynamics restricted to G, and we consider the measurable space $(G, \mathcal{T}', m')$, where $\mathcal{T}'$ is a σ-algebra of G over which m' is defined. We introduce G_{inv}, the "attractive" subset of G:

$$G_{inv} = \left\{x \in G \,|\, \exists t_0 > 0,\, \forall\, t > t_0,\, \phi^t x \in G\right\}.$$

If $x \in G_{inv}$, after some time it becomes possible to completely follow the trajectory starting from x. The assumption that we only have access to G is thus invisible concerning the dynamics on G_{inv}. Conversely, trajectories starting from $G\backslash G_{inv}$ leave G and then cannot be tracked. Fortunately, from Poincaré recurrence theorem, if $m(G) > 0$, then almost all trajectories come back into G. More precisely, we may define the Poincaré recurrence time as

$$\forall x \in G\backslash G_{inv},\ \tau_G(x) = \inf\left\{t > 0 \,|\, \phi^t x \in G,\, \exists t_0 \in (0,t),\, \phi^{t_0} x \notin G\right\}.$$

We remark that if $\tau_G(x) < \infty$, then $\phi^{\tau_G(x)} x \in \partial G \cap (G\backslash G_{inv})$, where ∂G is the boundary of G. Let G_{ext} be the set of the starting points of trajectories which never come back into G, i.e. points x such that $\tau_G(x) = \infty$:

$$G_{ext} = \left\{x \in G \,|\, \exists t_0 > 0,\, \forall\, t > t_0,\, \phi^t x \notin G\right\}.$$

Theorem 1 (Poincaré recurrence theorem)**.** *The set G_{ext} is negligible: $m(G_{ext}) = 0$.*

In that case, G is said m-recurrent. Until the end, if $m(G) = 0$, we suppose that G is m'-recurrent. We note G_{rec} the set of trajectories which alternatively wander inside and outside G:

$$G_{rec} = G\backslash\left(G_{inv} \cup G_{ext}\right).$$

We may remark that if $x \in G_{rec}$, then $\phi^{\tau_G(x)} x \in G_{rec}$. Although it is impossible to have a continuous description of the dynamics within G_{rec}, we may then track by intermittence the trajectories stemming from this set, thanks to recurrence times. Following Hilfer [11], we introduce the mapping

$$\begin{array}{cccc} S: & G_{rec} & \longrightarrow & G_{rec} \\ & x & \longmapsto & \phi^{\tau_G(x)} x. \end{array} \tag{2.1}$$

Iterations of S permit to follow the temporal evolution of a point of G_{rec}. Let us remark that for all $k \geq 1$, $S^k x \in \partial G$. Since $m'(G) = m'(G_{inv}) + m'(G_{rec})$, it is now possible to track almost all trajectories starting from G, at least by intermittence.

The following characteristic times will also be useful:

$$\forall x \in G_{rec}, \quad \tau_r(x) = \inf\left\{t > 0 \,|\, \phi^t x \notin G\right\}, \quad \tau_e(x) = \inf\left\{t > 0 \,|\, \phi^{\tau_r(x)+t} x \in G\right\}.$$

The time $\tau_r(x)$ is the time that the trajectory starting from x stays in G before leaving, while $\tau_e(x)$ is the time this trajectory then spends outside G (see figure 1). Those times verify

$$\tau_G(x) = \tau_r(x) + \tau_e(x).$$

We may also note that by continuity of the Hamiltonian flow, if $\tau_r(x) = 0$, then $x \in \partial G$.

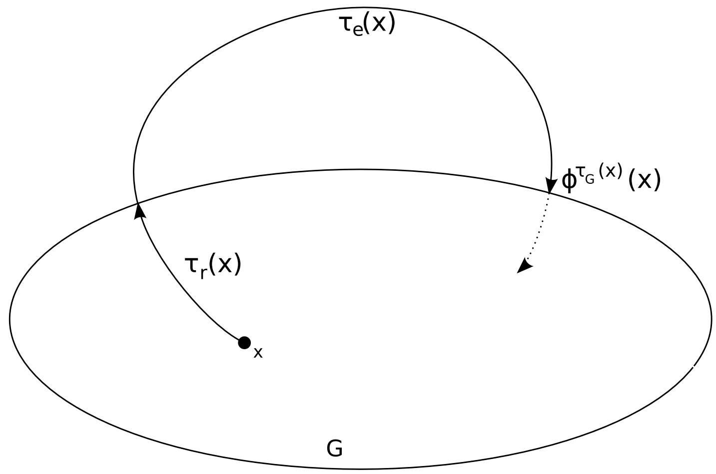

FIGURE 1. Characteristic times

In order to obtain global informations on the dynamics inside G, sets of trajectories - i.e. evolution of subsets of G - should be studied, for instance through the evolution of their measures.

This problem is studied in details in [11, 10]. The operator S is redefined at precision Δt on the set of measures on G and appears as a convolution product. For a measure ρ on G and a subset $A \subset G$, Hilfer obtains

$$S(\Delta t)\rho(A)(t) = p_{\Delta t} * \rho(A)(t).$$

Then he focuses on the induced dynamics after a renormalization in time scale (*continuous time-limit* in [11] and *ultra-long time limit* in [10]) and obtains a new operator associated to a new time step, $\tilde{S}(\widetilde{\Delta t})$. In that case, he shows that the characteristic derivative of this operator, more precisely the infinitesimal generator [8, p.356] $\mathcal{G}$ associated to the semi-group $\left(\tilde{S}(\widetilde{\Delta t})\right)_{\widetilde{\Delta t} \geq 0}$, defined by

$$\mathcal{G}\rho(A)(t) = \lim_{\widetilde{\Delta t} \to 0} \frac{\tilde{S}(\widetilde{\Delta t})\rho(A)(t) - \rho(A)(t)}{\widetilde{\Delta t}},$$

may be equal (up to the sign) to the Marchaud fractional derivative of order α [26, p.109], with $\alpha \in (0,1)$. Actually, this approach is part of a deeper questioning on time structure and irreversibility [14, 13].

However some points may still seem unclear, such as the signification of $S(\Delta t)$ and the renormalization procedures. Furthermore, the exponent α remains unspecified.

While keeping a similar approach, we propose here a simple dynamical model for which we study the infinitesimal generator. In several examples, it is proportional to the ordinary derivative. Then we use Zaslavsky analysis on Hamiltonian chaotic systems: it that case, the generator may become a fractional derivative.

3. CONSTRUCTION OF A SIMPLE MODEL

Let us recall our objective: we would like to describe the dynamics of our system restricted to G, in a global way, i.e. by considering sets of trajectories.

To do so, we introduce the mapping N given by

$$\begin{array}{rccc} N: & \mathcal{T}' & \longrightarrow & \mathcal{F}(\mathbb{R},\mathbb{R}) \\ & A & \longmapsto & N_A, \end{array}$$

where N_A is a real-valued function defined by

$$\begin{array}{rccc} N_A: & \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ & t & \longmapsto & m'((\phi^t A)\cap G). \end{array}$$

Let $A \in \mathcal{T}'$ and $t_0 \in \mathbb{R}$. We want to know the infinitesimal evolution of $N_A(t_0)$. Several successive steps will lead us to a general formula.

3.1. **Model with one trap.** Let $\Delta t > 0$. We consider the following binary dynamics: all of the trajectories which leave G are "trapped" within $P \subset \Gamma$, with $P \cap G = \emptyset$. Then they come back after $2\Delta t$, and stay in G during a multiple of Δt, until possibly leaving again.

We may then split G with the two following sets:

$$G_0(\Delta t) = \{x \in G \,|\, \tau_r(x) \geq \Delta t\}, \tag{3.1}$$

$$G_1(\Delta t) = \{x \in G \,|\, \tau_r(x) < \Delta t,\, \tau_e(x) = 2\Delta t\}. \tag{3.2}$$

We remark that $G_1(\Delta t)$ may also be written as

$$\begin{aligned} G_1(\Delta t) &= \{x \in G \,|\, \tau_r(x) = 0,\, \tau_e(x) = 2\Delta t\}, \\ &= \{x \in G \,|\, \tau_r(x) < \Delta t\}, \\ &= \{x \in G \,|\, \tau_r(x) = 0\}. \end{aligned}$$

This set is directly linked to trap P.

Hence we have $G_0(\Delta t) \cap G_1(\Delta t) = \emptyset$ and $G = G_0(\Delta t) \cup G_1(\Delta t)$.

As in [10], we define the numbers

$$p_k(\Delta t) = \frac{m'(G_k(\Delta t))}{m'(G)}, \quad k \in \{0,1\}.$$

Those two quantities provide a probability density associated to recurrence times ($p_0(\Delta t) + p_1(\Delta t) = 1$).

We suppose that these sets are "well mixed":

$$\forall B \in \mathcal{T}',\; m'(B \cap G_0(\Delta t)) = p_0(\Delta t) m'(B),\; m'(B \cap G_1(\Delta t)) = p_1(\Delta t) m'(B).$$

Starting from $N_A(t_0) = m'(A)$, we determine the following states. The shifts will occur every Δt, so we may just consider $N_A(t_0 + n\Delta t)$, with $n \in \mathbb{N}$. Those successive instants are now detailed.

(1) At t_0^+, trajectories starting from A are splitting: some of them stay in G while the others leave G during $2\Delta t$. We note A_0 the set of initial conditions of the first ones and A_1 the set of the second ones. Consequently, $m'(A_0) = p_0 m'(A)$ and $m'(A_1) = p_1 m'(A)$ (we omit the dependance of p_0 and p_1 in Δt).

(2) At $t_0 + \Delta t$, only A_0 is in G:

$$N_A(t_0 + \Delta t) = m'(A_0) = p_0 N_A(t_0).$$

Within the trap, A_1 becomes A_{11}.

At $t_0^+ + \Delta t$, it is now A_0 which splits similarly to A, and gives birth to A_{00} and A_{01}: $m'(A_{00}) = p_0 m'(A_0)$ and $m'(A_{01}) = p_1 m'(A_0)$.

(3) Trajectories which escaped from G at t_0^+ come back at $t_0^+ + 2\Delta t$. Consequently, at $t_0 + 2\Delta t$, only A_{00} is present in G:

$$N_A(t_0 + 2\Delta t) = m'(A_{00}) = p_0 N_A(t_0 + \Delta t).$$

At $t_0^+ + 2\Delta t$, A_{00} splits into A_{000} and A_{001}, A_{11} comes back (it turns into A_{110}), and A_{01} stays outside G while becoming A_{011}.

(4) At $t_0 + 3\Delta t$, G contains A_{000} and A_{110}. Hence we have

$$\begin{aligned} N_A(t_0 + 3\Delta t) &= m'(A_{000}) + m'(A_{110}), \\ &= p_0 m'(A_{00}) + m'(A_1), \\ &= p_0 N_A(t_0 + 2\Delta t) + p_1 N_A(t_0). \end{aligned}$$

At $t_0^+ + 3\Delta t$, A_{000} splits into A_{0000} and A_{0001}, A_{110} into A_{1100} and A_{1101}, A_{011} comes back and becomes A_{0110}. Concerning A_{001}, it stays outside G and turns into A_{0011}.

(5) At $t_0 + 4\Delta t$, we find in G the sets A_{0000}, A_{1100} and A_{0110}:

$$\begin{aligned} N_A(t_0 + 4\Delta t) &= m'(A_{0000}) + m'(A_{1100}) + m'(A_{0110}), \\ &= p_0 \left(m'(A_{000}) + m'(A_{110}) \right) + m'(A_{01}), \\ &= p_0 N_A(t_0 + 3\Delta t) + p_1 N_A(t_0 + \Delta t). \end{aligned}$$

A sketch of the dynamics is given in figure 2.

An immediate generalization leads to

$$\forall\, n \in \mathbb{Z},\; N_A(t_0 + n\Delta t) = p_0 N_A(t_0 + (n-1)\Delta t) + p_1 N_A(t_0 + (n-3)\Delta t).$$

In particular,

$$N_A(t_0 + \Delta t) = p_0 N_A(t_0) + p_1 N_A(t_0 - 2\Delta t). \tag{3.3}$$

Keeping in mind definition (2.1), we note $S(\Delta t)$ the operator of “infinitesimal” temporal evolution, which leads roughly speaking to the next temporal step. In this example, time is discete and takes its values in $t_0 + \Delta t\mathbb{Z} + 2\Delta t\mathbb{Z} = t_0 + \Delta t\mathbb{Z}$.

So we have

$$S(\Delta t) N_A(t_0) = N_A(t_0 + \Delta t). \tag{3.4}$$

Given that $\lim\limits_{\Delta t \to 0^+} p_0 N_A(t_0) + p_1 N_A(t_0 - 2\Delta t) = N_A(t_0)$, $S(\Delta t)$ verifies

$$S(0) = \text{ id}. \tag{3.5}$$

Moreover, from (3.4) $S(\Delta t)$ also verifies

$$\forall \Delta t_1,\, \Delta t_2 > 0,\; S(\Delta t_1)\, S(\Delta t_2) = S(\Delta t_1 + \Delta t_2). \tag{3.6}$$

Let us remark that (3.3) may not used to check this property. By verifying (3.5) and (3.6), $(S(\Delta t))_{\Delta t \geq 0}$ defines a one-parameter semi-group. If N_A possesses a left derivative at t_0, the associated infinitesimal generator $\mathcal{G}$ is given by

$$\mathcal{G} N_A(t_0) = \lim_{\Delta t \to 0^+} \frac{S(\Delta t) N_A(t_0) - N_A(t_0)}{\Delta t}, \tag{3.7}$$

$$= \lim_{\Delta t \to 0^+} \frac{p_1(\Delta t) \left(N_A(t_0 - 2\Delta t) - N_A(t_0) \right)}{\Delta t}, \tag{3.8}$$

$$= -2 p_1(0^+) \frac{d}{dt^-} N_A(t_0), \tag{3.9}$$

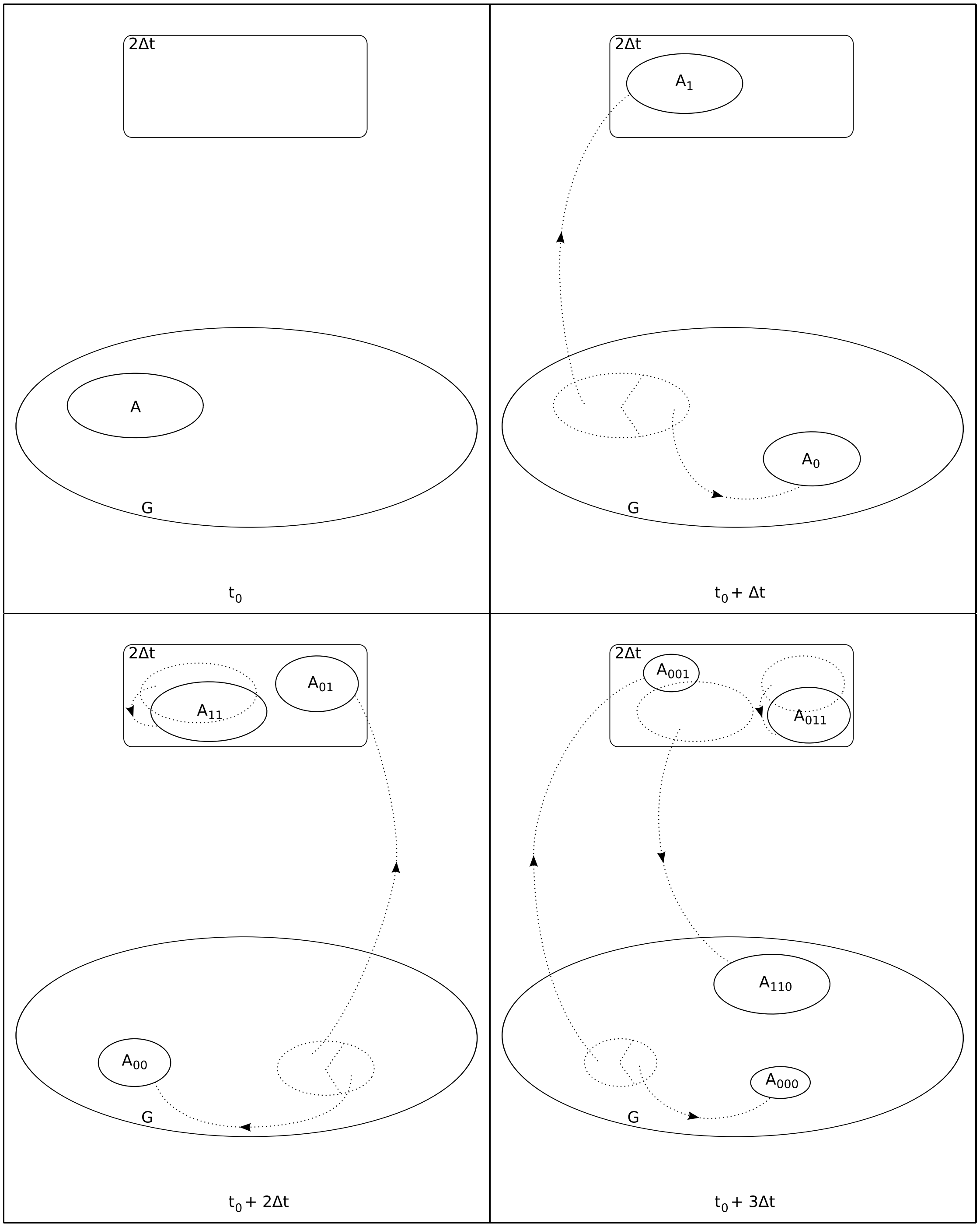

FIGURE 2. Model with one trap

where $\frac{d}{dt^-}N_A(t_0)$ is the left derivative of N_A at t_0 and $p_1(0^+)$ is the right limit of p_1 at 0.

Remark 1. *In this example, the function N_A cannot be differentiable at t_0, unless $N_A'(t_0) = 0$. Indeed, $S(\Delta t)N_A(t_0) = N_A(t_0 + \Delta t)$ in that case, so we also have $\mathcal{G}N_A(t_0) = \frac{d}{dt^+}N_A(t_0)$.*

3.2. **Model with two traps.** We generalize the previous example by supposing that there are now two sets P_1 et P_2 outside G, which "trap" trajectories during $2\Delta t$ and $3\Delta t$ respectively. Trapped trajectories then stay in G during a multiple of Δt.

As previously, we introduce the following sets:

$$\begin{aligned} G_0(\Delta t) &= \{x \in G \,|\, \tau_r(x) \geq \Delta t\}, \\ G_1(\Delta t) &= \{x \in G \,|\, \tau_r(x) < \Delta t,\, \tau_e(x) = 2\Delta t\}, \\ G_2(\Delta t) &= \{x \in G \,|\, \tau_r(x) < \Delta t,\, \tau_e(x) = 3\Delta t\}. \end{aligned}$$

Once again, those sets form a partition of G. For $k \in \{0, 2\}$, we note $p_k(\Delta t) = \dfrac{m'(G_k(\Delta t))}{m'(G)}$. We still have $p_0(\Delta t) + p_1(\Delta t) + p_2(\Delta t) = 1$.

By proceeding similaraly to the previous model, we find:

$$N_A(t_0 + \Delta t) = p_0 N_A(t_0) + p_1 N_A(t_0 - 2\Delta t) + p_2 N_A(t_0 - 3\Delta t).$$

Time evolves here in $t_0 + \Delta t\mathbb{Z} + 2\Delta t\mathbb{Z} + 3\Delta t\mathbb{Z} = t_0 + \Delta t\mathbb{Z}$. The infinitesimal evolution operator $S(\Delta t)$ once again verifies

$$S(\Delta t)N_A(t_0) = N_A(t_0 + \Delta t),$$

thus semi-group properties (3.5) and (3.6) are still fulfilled.

Concerning the infinitesimal generator, we have

$$\begin{aligned} \mathcal{G}N_A(t_0) &= \lim_{\Delta t \to 0^+} \frac{p_1\left(N_A(t_0 - 2\Delta t) - N_A(t_0)\right)}{\Delta t} + \frac{p_2\left(N_A(t_0 - 3\Delta t) - N_A(t_0)\right)}{\Delta t}, \\ &= -(2p_1(0^+) + 3p_2(0^+))\frac{d}{dt^-}N_A(t_0). \end{aligned}$$

3.3. **Generalizations.** Let $\{P_k\}_{k \in \mathbb{N}^*}$ be a set of traps with respective trapping times $n_k\Delta t$, $n_k \in \mathbb{N}^*$. We assume that each time a trajectory leaves G, it is trapped by exactly one trap. Hence, if we note

$$G_0(\Delta t) = \{x \in G \,|\, \tau_r(x) \geq \Delta t\}$$

and for all $k \in \mathbb{N}^*$,

$$G_k(\Delta t) = \{x \in G \,|\, \tau_r(x) < \Delta t,\, \tau_e(x) = n_k\Delta t\},$$

we still obtain a partition of G.

For all $k \in \mathbb{N}$, if we set

$$p_k(\Delta t) = \frac{m'(G_k(\Delta t))}{m'(G)}, \tag{3.10}$$

the evolution of N_A verifies

$$N_A(t_0 + \Delta t) = \sum_{k \geq 0} p_k(\Delta t) N_A(t_0 - n_k\Delta t). \tag{3.11}$$

Once again, the successive instants belong to $t_0 + \Delta t\mathbb{Z}$. Then $S(\Delta t)N_A(t_0) = N_A(t_0 + \Delta t)$, and

$$S(\Delta t)N_A(t_0) = \sum_{k \geq 0} p_k(\Delta t)N_A(t_0 - n_k\Delta t). \tag{3.12}$$

Now we consider any trapping times, denoted $T_k(\Delta t)$ with $k \in \mathbb{N}^*$, and we suppose they are well-ordered:

$$0 < T_1(\Delta t) < \cdots < T_k(\Delta t) < \cdots .$$

In that case, the group $\sum_{k\geq 1} T_k(\Delta t)\mathbb{Z}$ cannot anymore be written as $\tau_0\mathbb{Z}$ but is dense in $\mathbb{R}$. In particular, a minimal time step cannot anymore be defined. But the group $\sum_{k\geq 1} T_k(\Delta t)\mathbb{Z}$ remains countable, so it is still possible to move to the next step: the operator $S(\Delta t)$ still makes sense, but is no longer equal to $N_A(t_0 + \Delta t)$. Consequently, a generalization cannot be done with (3.11), but with (3.12):

$$S(\Delta t)N_A(t_0) = \sum_{k \geq 0} p_k(\Delta t)N_A(t_0 - T_k(\Delta t)), \tag{3.13}$$

where $p_k(\Delta t)$ is given by (3.10), with

$$G_0(\Delta t) = \{x \in G \,|\, \tau_r(x) \geq T_1(\Delta t)\}\,, \tag{3.14}$$

and, for all $k \in \mathbb{N}^*$,

$$G_k(\Delta t) = \{x \in G \,|\, \tau_r(x) < T_1(\Delta t),\, \tau_e(x) = T_k(\Delta t)\}\,. \tag{3.15}$$

This formula is to be linked with expression (8) in [10].

We assume by now that $\Delta t \mapsto p_0(\Delta t)$ has a right limit at 0, denoted $p_0(0^+)$. Now we specify values of $p_k(\Delta t)$ and $T_k(\Delta t)$ in the case of chaotic Hamiltonian systems.

4. Dynamical traps and anomalous diffusion

Zaslavsky studies in [33] the general shape of chaotic Hamiltonian phase spaces. In chapter 12, he introduces the notion of *dynamical trap* so as to describe the behavior of trajectories near KAM tori. This area possesses a self-similar structure: it is composed of imbricated subsets P_k which verify

$$m(P_{k+1}) = \lambda_M\, m(P_k), \text{ with } \lambda_M < 1.$$

Moreover, the trapping times T_k associated also have a self-similar property:

$$T_{k+1} = \lambda_T\, T_k, \text{ with } \lambda_T > 1.$$

Trapping times are hence all the longer as traps are small. This analysis can also be found in [32, 34].

The kind of structure has “macroscopic” consequences: when one studies diffusion of particules through a probabilistic descritpion of the system, the moment of order 2 is ruled by the following law:

$$\langle x^2 \rangle \propto t^\mu.$$

The classical case (normal diffusion) corresponds to $\mu = 1$. The terms subdiffusion and superdiffusion are respectively used for $\mu < 1$ and $\mu > 1$. See [33, part.3] and [28] for more details.

Those anomalous diffusion phenomena can be described with the introduction of fractional derivatives into some specific partial derivatives equations [20, 31, 25], [33, chap. 16].

One of the equations proposed by Zaslavsky [33, p.249], a simplified fractional Fokker-Planck-Kolmogorov equation, is given by

$$\frac{\partial^\beta}{\partial t^\beta} P(x,t) = \frac{\partial^\alpha}{\partial x^\alpha} \left(\mathcal{A}(x) P(x,t) \right), \; 0 < \beta \leq 1, \; 0 < \alpha \leq 2, \tag{4.1}$$

where $P(x,t)$ is the probability to find the particule at position x at time t.

If we assume $\mathcal{A}$ constant, this equation leads to the following transport equation [33, p.251]:

$$\langle x^\alpha \rangle \propto t^\beta.$$

The classical case corresponds to $\beta = 1$ and $\alpha = 2$. The transport exponent [33, p.192] is defined by

$$\mu = \frac{2\beta}{\alpha}. \tag{4.2}$$

According to Zaslavsky [33, p.251, p.263], the influence of the dynamical traps appears through the following relation:

$$\mu = \frac{|\ln(\lambda_M)|}{\ln(\lambda_T)}. \tag{4.3}$$

Equality between (4.2) and (4.3) provides a connection between the fractal structure of the phase space and the fractional derivatives of (4.1). However, the justification for the introduction of those derivatives in equations (16.3) and (16.4) of [33] is not clear. An approach based on Continuous Time Random Walks (CTRW) [18, 20, 30] leads to such derivatives, but those probabilistic models do not rely on the "microscopic" dynamics of the trajectories.

We propose here to link the emergence of fractional operators with the self-similar structure of the phase space described above.

Coefficients λ_M and λ_T *a priori* depend on Δt. Because of the dynamical definition of traps P_k, the subsets $G_k(\Delta t)$ also verify, for $k \geq 1$,

$$m'(G_{k+1}(\Delta t)) = \lambda_M(\Delta t)\, m'(G_k(\Delta t)).$$

Concerning the characteristic times, we have, for all $k \geq 1$,

$$T_k(\Delta t) = T_1(\Delta t)\, \lambda_T(\Delta t)^{k-1}.$$

We would like that the structure of the traps becomes thiner when $\Delta t \to 0$, while remaining self-similar. This leads us to assume

$$\lambda_M(\Delta t) = (\lambda_M)^{\Delta t} \;\text{ and } \lambda_T(\Delta t) = (\lambda_T)^{\Delta t}, \tag{4.4}$$

where by sake of lisibility, λ_M and λ_T are now two real numbers such that $0 < \lambda_M < 1$ and $\lambda_T > 1$.

Remark 2. *The transport exponent remains unchanged with definition* (4.4)*:*

$$\forall \Delta t > 0, \; \mu = \frac{|\ln(\lambda_M(\Delta t))|}{\ln(\lambda_T(\Delta t))} = \frac{|\ln(\lambda_M)|}{\ln(\lambda_T)}.$$

Consequently, for all $k \geq 1$,

$$m'(G_k(\Delta t)) = (\lambda_M)^{(k-1)\Delta t}\, m'(G_1(\Delta t)),$$

and

$$T_k(\Delta t) = T_1(\Delta t)\, \lambda_T^{(k-1)\Delta t}. \tag{4.5}$$

In order to obtain smaller characteristic times when $\Delta t \to 0$, we suppose that

$$\lim_{\Delta t \to 0^+} T_1(\Delta t) = 0. \tag{4.6}$$

Using relation $\sum_{k\geq 0} p_k(\Delta t) = 1$, we find that for all $k \geq 1$,

$$p_k(\Delta t) = (1 - p_0(\Delta t))(1 - \lambda_M^{\Delta t})\lambda_M^{(k-1)\Delta t}. \tag{4.7}$$

The infinitesimal evolution (3.13) of the system thus becomes

$$S(\Delta t)N_A(t_0) = p_0(\Delta t)N_A(t_0) + (1 - p_0(\Delta t))\,(1 - \lambda_M^{\Delta t}) \sum_{k\geq 0} \lambda_M^{k\Delta t} N_A\left(t_0 - T_1(\Delta t)\lambda_T^{k\Delta t}\right). \tag{4.8}$$

The infinitesimal generators related to (4.8) can now be determined.

5. Fractional infinitesimal generator

Hölder conditions on N_A appear in this part, so we need the following definitions. Let $\Omega \subset \mathbb{R}$, $f : \Omega \to \mathbb{R}$ and $\alpha \in (0, 1]$.

Definition 1. *Let $x \in \Omega$. The function f satisfies the Hölder condition of order α at x if*

$$\exists c > 0,\ \exists \eta > 0,\ \forall\, y \in \Omega,\ |x - y| \leq \eta \Rightarrow |f(x) - f(y)| \leq c|x - y|^\alpha.$$

Definition 2. *The function f locally satisfies the Hölder condition of order α if for all $x \in \Omega$, f satisfies the Hölder condition of order α at x.*

Definition 3. *The function f satisfies the Hölder condition of order α if*

$$\exists c > 0,\ \forall\, x,\, y \in \Omega,\ |f(x) - f(y)| \leq c|x - y|^\alpha.$$

If $\alpha = 1$, f is called Lipschitz continuous.

Now we go back to our problem and we begin to show that $S(\Delta t)$ still fulfills (3.5).

Lemma 1. *If N_A satisfies the Hölder condition of order β, with $\beta < \mu$, then*

$$\lim_{\Delta t \to 0^+} S(\Delta t)N_A(t_0) = N_A(t_0).$$

Proof. The difference $S(\Delta t)N_A(t_0) - N_A(t_0)$ verifies

$$S(\Delta t)N_A(t_0) - N_A(t_0) = (1 - p_0(\Delta t))\,(1 - \lambda_M^{\Delta t}) \sum_{k\geq 0} \lambda_M^{k\Delta t} \left[N_A\left(t_0 - T_1(\Delta t)\lambda_T^{k\Delta t}\right) - N_A(t_0)\right].$$

We remark that $\lambda_M \lambda_T^\beta < 1$ if and only if $\beta < \mu$. Given that N_A satisfies the Hölder condition of order β, we obtain

$$\begin{aligned} |S(\Delta t)N_A(t_0) - N_A(t_0)| &\leq (1 - p_0(\Delta t))\,(1 - \lambda_M^{\Delta t}) \sum_{k\geq 0} \lambda_M^{k\Delta t} \left(T_1(\Delta t)\lambda_T^{k\Delta t}\right)^\beta \\ &\leq (1 - p_0(\Delta t))\, T_1(\Delta t)^\beta \frac{1 - \lambda_M^{\Delta t}}{1 - \left(\lambda_M \lambda_T^\beta\right)^{\Delta t}} \end{aligned}$$

On the one hand,

$$\lim_{\Delta t \to 0^+} (1 - p_0(\Delta t))\, \frac{1 - \lambda_M^{\Delta t}}{1 - \left(\lambda_M \lambda_T^\beta\right)^{\Delta t}} = (1 - p_0(0^+)) \frac{\ln(\lambda_M)}{\ln(\lambda_M \lambda_T^\beta)},$$

and on the other hand, $\lim_{\Delta t \to 0^+} T_1(\Delta t) = 0$ from assumption (4.6).

Consequently, $\lim_{\Delta t \to 0^+} (S(\Delta t)N_A(t_0) - N_A(t_0)) = 0$.

□

As it has already be seen, definition (3.3) cannot be used to check property (3.6). So we just assume that (3.6) is fulfilled.

We recall that the infinitesimal generator $\mathcal{G}$ associated to this semi-group verifies

$$\mathcal{G}N_A(t_0) = \lim_{\Delta t \to 0^+} \frac{S(\Delta t)N_A(t_0) - N_A(t_0)}{\Delta t}.$$

In [14, 10, 13, 11], Hilfer shows that fractional derivatives may appear as infinitesimal generators of renormalized evolution operators. A similar result will now be obtained, but without using any renormalization.

For all $\Delta t > 0$, we note

$$\mathcal{G}(\Delta t)N_A(t_0) = \frac{S(\Delta t)N_A(t_0) - N_A(t_0)}{\Delta t}.$$

We also introduce the function f defined by

$$\begin{array}{rccc} f\ : & \mathbb{R}^+ \times \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & (\Delta t, y) & \longmapsto & \lambda_M^y \left[N_A \left(t_0 - T_1(\Delta t)\lambda_T^y \right) - N_A(t_0) \right]. \end{array}$$

Consequently,

$$\mathcal{G}(\Delta t)N_A(t_0) = (1 - p_0(\Delta t))\, \frac{1 - \lambda_M^{\Delta t}}{\Delta t} \sum_{k \geq 0} f(\Delta t, k\Delta t).$$

For all $k \in \mathbb{N}$, we note

$$\begin{aligned} I_k(\Delta t) &= f(\Delta t, k\Delta t) = \int_k^{k+1} f(\Delta t, k\Delta t)\, dx, \\ J_k(\Delta t) &= \int_k^{k+1} f(\Delta t, x\Delta t)\, dx. \end{aligned}$$

Hence $\mathcal{G}(\Delta t)N_A(t_0)$ can be written as

$$\mathcal{G}(\Delta t)N_A(t_0) = (1 - p_0(\Delta t))\, \frac{1 - \lambda_M^{\Delta t}}{\Delta t} \sum_{k \geq 0} I_k(\Delta t).$$

5.1. **Case $\mu > 1$.** In that case, $\lambda_M \lambda_T < 1$.

Theorem 2. *If N_A is differentiable and Lipschitz continuous on $\mathbb{R}$, and if T_1 is differentiable at 0, then*

$$\mathcal{G}N_A(t_0) = -\gamma\, \frac{d}{dt} N_A(t_0),$$

where $\gamma = (1 - p_0(0^+))\, \dfrac{\mu}{\mu - 1} T_1'(0)$.

Proof. First we prove that $\lim_{\Delta t \to 0^+} \sum_{k \geq 0} (I_k(\Delta t) - J_k(\Delta t)) = 0$.

The function N_A is differentiable, so is $y \mapsto f(\Delta t, y)$ for all $\Delta t \geq 0$, and

$$\partial_2 f(\Delta t, y) = \ln(\lambda_M) f(\Delta t, y) - T_1(\Delta t)\, (\lambda_M \lambda_T)^y\, N_A'(t_0 - T_1(\Delta t)\lambda_T^y).$$

If we note c the Lipschitz constant, we have

$$|\partial_2 f(\Delta t, y)| \leq \ln(\lambda_M) c T_1(\Delta t) \left(\lambda_M \lambda_T\right)^y + c T_1(\Delta t) \left(\lambda_M \lambda_T\right)^y.$$

By setting $c' = c(1 + \ln(\lambda_T))$, we obtain

$$|\partial_2 f(\Delta t, y)| \leq c' T_1(\Delta t) \left(\lambda_M \lambda_T\right)^y.$$

Let $k \in \mathbb{N}$. Then

$$\begin{aligned}
|I_k(\Delta t) - J_k(\Delta t)| &\leq \int_k^{k+1} |f(\Delta t, x\Delta t) - f(\Delta t, k\Delta t)|\, dx, \\
&\leq \Delta t \sup_{[k,k+1]} |\partial_2 f(\Delta t, x\Delta t)|, \\
&\leq c' \Delta t T_1(\Delta t) \left(\lambda_M \lambda_T\right)^{k\Delta t}.
\end{aligned}$$

Given that $\lim_{\Delta t \to 0^+} \Delta t \sum_{k \geq 0} \left(\lambda_M \lambda_T\right)^{k\Delta t} = \frac{1}{|\ln(\lambda_M \lambda_T)|}$ and $\lim_{\Delta t \to 0^+} T_1(\Delta t) = 0$, we infer that

$$\lim_{\Delta t \to 0^+} \sum_{k \geq 0} \left(I_k(\Delta t) - J_k(\Delta t)\right) = 0.$$

Consequently,

$$\mathcal{G}(\Delta t) N_A(t_0) \underset{\Delta t \to 0^+}{\sim} -\left(1 - p_0(0^+)\right) \ln(\lambda_M) \sum_{k \geq 0} J_k(\Delta t). \tag{5.1}$$

Now we can evaluate $\lim_{\Delta t \to 0^+} \sum_{k \geq 0} J_k(\Delta t)$. Firstly,

$$\sum_{k \geq 0} J_k(\Delta t) = \int_0^\infty \lambda_M^{x\Delta t} \left[N_A\left(t_0 - T_1(\Delta t) \lambda_T^{x\Delta t}\right) - N_A(t_0)\right] dx.$$

With substitution $t = \lambda_T^{x\Delta t}$, we obtain

$$\begin{aligned}
\sum_{k \geq 0} J_k(\Delta t) &= \frac{1}{\Delta t \ln(\lambda_T)} \int_1^\infty t^{-(1+\mu)} \left[N_A(t_0 - tT_1(\Delta t)) - N_A(t_0)\right] dt, \qquad (5.2) \\
&= \frac{1}{\ln(\lambda_T)} \frac{T_1(\Delta t)}{\Delta t} \int_1^\infty t^{-\mu} \frac{N_A(t_0 - tT_1(\Delta t)) - N_A(t_0)}{tT_1(\Delta t)}\, dt.
\end{aligned}$$

For all $t \geq 1$, $\left|t^{-\mu} \frac{N_A(t_0 - tT_1(\Delta t)) - N_A(t_0)}{tT_1(\Delta t)}\right| \leq ct^{-\mu}$, and $t \mapsto ct^{-\mu}$ is integrable on $[1, +\infty)$. Moreover,

$$\lim_{\Delta t \to 0^+} \frac{N_A(t_0 - tT_1(\Delta t)) - N_A(t_0)}{tT_1(\Delta t)} = -N_A'(t_0).$$

By dominated convergence,

$$\begin{aligned}
\lim_{\Delta t \to 0^+} \sum_{k \geq 0} J_k(\Delta t) &= \frac{1}{\ln(\lambda_T)} T_1'(0)(-N_A'(t_0)) \int_1^\infty t^{-\mu}\, dt, \\
&= \frac{1}{(1-\mu)\ln(\lambda_T)} T_1'(0) N_A'(t_0).
\end{aligned}$$

Finally, from (5.1),

$$\lim_{\Delta t\to 0^+} \mathcal{G}(\Delta t) N_A(t_0) = -\left(1-p_0(0^+)\right) \frac{\ln(\lambda_M)}{(1-\mu)\ln(\lambda_T)} T_1'(0) N_A'(t_0),$$
$$= -\left(1-p_0(0^+)\right) \frac{\mu}{\mu-1} T_1'(0) N_A'(t_0).$$

□

Remark 3. *If N_A verifies assumptions of theorem 2, then N_A satisfies the Hölder condition of order 1 and consequently fulfills conditions of lemma 1.*

5.2. **Case $\mu < 1$.** We can try to estimate $\lim_{\Delta t\to 0^+} \sum_{k\geq 0} J_k(\Delta t)$, assuming that N_A is smooth enough and rapidly decreasing in $-\infty$, in order that all the following quantities are well-defined. We integrate by parts (5.2):

$$\sum_{k\geq 0} J_k(\Delta t) = \frac{-T_1(\Delta t)}{\mu \Delta t \ln(\lambda_T)} \int_1^\infty t^{-\mu} N_A'(t_0 - tT_1(\Delta t))\, dt + \frac{N_A(t_0 - T_1(\Delta t)) - N_A(t_0)}{\mu \Delta t \ln(\lambda_T)}\, dt.$$

Substitution $u = T_1(\Delta t)t$ leads to

$$\sum_{k\geq 0} J_k(\Delta t) = \frac{-T_1(\Delta t)^\mu}{\mu \Delta t \ln(\lambda_T)} \int_{T_1(\Delta t)}^\infty u^{-\mu} N_A'(t_0 - u)\, du + \frac{N_A(t_0 - T_1(\Delta t)) - N_A(t_0)}{\mu \Delta t \ln(\lambda_T)}\, du.$$

The integral is not problematic:

$$\lim_{\Delta t\to 0^+} \int_{T_1(\Delta t)}^\infty u^{-\mu} N_A'(t_0 - u)\, du = \int_0^\infty u^{-\mu} N_A'(t_0 - u)\, du.$$

If T_1 is differentiable at 0, then

$$\lim_{\Delta t\to 0^+} \frac{N_A(t_0 - T_1(\Delta t)) - N_A(t_0)}{\mu \Delta t \ln(\lambda_T)} = -\frac{T_1'(0)}{\mu \ln(\lambda_T)} N_A'(t_0).$$

Conversely, $\lim_{\Delta t\to 0^+} \dfrac{T_1(\Delta t)^\mu}{\Delta t} = +\infty$. So we cannot find any infinitesimal generator. The assumption on the differentiability of T_1 at 0 should hence be replaced.

In order that $\dfrac{T_1(\Delta t)^\mu}{\Delta t}$ has a finite limit, we suppose that there exists $b > 0$ such that

$$T_1(\Delta t) \underset{\Delta t\to 0^+}{\sim} b\,(\Delta t)^{1/\mu}.$$

From a physical point of view, $T_1(\Delta t)$ and Δt are homogeneous to time, so we introduce a constant of time τ such that $b = \tau^{1-1/\mu}$:

$$T_1(\Delta t) \underset{\Delta t\to 0^+}{\sim} \tau^{1-1/\mu} (\Delta t)^{1/\mu}. \tag{5.3}$$

Under this assumption on T_1, a fractional derivative defined as follows will appear.

Definition 4. *Let $f : \mathbb{R} \to \mathbb{R}$ and $\alpha \in (0,1)$. The Marchaud fractional derivative of order α is defined as*

$$\mathcal{D}_+^\alpha f(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty u^{-(1+\alpha)} \left[f(t) - f(t-u)\right] du,$$

where Γ is the Gamma function.

This derivative is well-defined if f is bounded and locally satisfies the Hölder condition of order δ, with $\delta > \alpha$. See [26, p.109] for more details.

Now we can enunciate the main result of the article.

Theorem 3. *If N_A satisfies the Hölder condition of order β and locally satisfies the Hölder condition of order ν, with $\beta < \mu < \nu$, then*

$$\mathcal{G}N_A(t_0) = -\tilde{\gamma}\,\tau^{\mu-1}\,\mathcal{D}_+^{\mu}\,N_A(t_0), \tag{5.4}$$

where $\tilde{\gamma} = \Gamma(1-\mu)(1-p_0(0^+))$.

Proof. As previously, we firstly prove that $\lim_{\Delta t\to 0^+} \sum_{k\geq 0} (I_k(\Delta t) - J_k(\Delta t)) = 0$.

Let $k \in \mathbb{N}$. For all $x \in [k, k+1]$,

$$\begin{aligned}
f(\Delta t, x\Delta t) - f(\Delta t, k\Delta t) &= \lambda_M^{x\Delta t}\left[\left(N_A(t_0 - T_1(\Delta t)\lambda_T^{x\Delta t}) - N_A(t_0)\right)\right.\\
&\qquad\left. - \left(N_A(t_0 - T_1(\Delta t)\lambda_T^{k\Delta t}) - N_A(t_0)\right)\right]\\
&\qquad + \left[\lambda_M^{x\Delta t} - \lambda_M^{k\Delta t}\right]\left(N_A(t_0 - T_1(\Delta t)\lambda_T^{k\Delta t}) - N_A(t_0)\right),\\
&= \lambda_M^{x}\left(N_A(t_0 - T_1(\Delta t)\lambda_T^{x\Delta t}) - N_A(t_0 - T_1(\Delta t)\lambda_T^{k\Delta t})\right)\\
&\qquad + \left[\lambda_M^{x\Delta t} - \lambda_M^{k\Delta t}\right]\left(N_A(t_0 - T_1(\Delta t)\lambda_T^{k\Delta t}) - N_A(t_0)\right).
\end{aligned}$$

Concerning the first right-hand member, we obtain the following inequality:

$$\begin{aligned}
&\left|\int_k^{k+1} \lambda_M^{x\Delta t}\left(N_A(t_0 - T_1(\Delta t)\lambda_T^{x\Delta t}) - N_A(t_0 - T_1(\Delta t)\lambda_T^{k\Delta t})\right)\,dx\right|\\
&\qquad \leq \lambda_M^{k\Delta t}\int_k^{k+1}\left|N_A(t_0 - T_1(\Delta t)\lambda_T^{x\Delta t}) - N_A(t_0 - T_1(\Delta t)\lambda_T^{k\Delta t})\right|dx,\\
&\qquad \leq \lambda_M^{k\Delta t}\int_k^{k+1}\left|T_1(\Delta t)\left(\lambda_T^{x\Delta t} - \lambda_T^{k\Delta t}\right)\right|^{\beta}dx,\\
&\qquad \leq \lambda_M^{k\Delta t}T_1(\Delta t)^{\beta}\left(\lambda_T^{(k+1)\Delta t} - \lambda_T^{k\Delta t}\right)^{\beta},\\
&\qquad \leq T_1(\Delta t)^{\beta}\left(\lambda_T^{\Delta t} - 1\right)\left(\lambda_M\lambda_T^{\beta}\right)^{k\Delta t}.
\end{aligned}$$

For the second one, we have

$$\begin{aligned}
&\left|\int_k^{k+1}\left[\lambda_M^{x\Delta t} - \lambda_M^{k\Delta t}\right]\left(N_A(t_0 - T_1(\Delta t)\lambda_T^{k\Delta t}) - N_A(t_0)\right)\,dx\right|\\
&\qquad \leq T_1(\Delta t)^{\beta}\lambda_T^{k\beta\Delta t}\int_k^{k+1}\left|\lambda_M^{x\Delta t} - \lambda_M^{k\Delta t}\right|dx,\\
&\qquad \leq T_1(\Delta t)^{\beta}\lambda_T^{k\beta\Delta t}\left(\lambda_M^{(k+1)\Delta t} - \lambda_M^{k\Delta t}\right),\\
&\qquad \leq T_1(\Delta t)^{\beta}\left(\lambda_M^{\Delta t} - 1\right)\left(\lambda_M\lambda_T^{\beta}\right)^{k\Delta t}.
\end{aligned}$$

Consequently,

$$|I_k(\Delta t) - J_k(\Delta t)| \leq T_1(\Delta t)^{\beta}\left(\lambda_T^{\Delta t} - 1 + \lambda_M^{\Delta t} - 1\right)\left(\lambda_M\lambda_T^{\beta}\right)^{k\Delta t}.$$

Since $\beta < \mu$, $\lambda_M \lambda_T^\beta < 1$. Thus,

$$\sum_{k \geq 0} |I_k(\Delta t) - J_k(\Delta t)| \leq T_1(\Delta t)^\beta \frac{\lambda_T^{\Delta t} - 1 + \lambda_M^{\Delta t} - 1}{1 - \lambda_M \lambda_T^\beta}.$$

Given that $\lim\limits_{\Delta t \to 0^+} \dfrac{\lambda_T^{\Delta t} - 1 + \lambda_M^{\Delta t} - 1}{1 - \lambda_M \lambda_T^\beta} = -\dfrac{\ln(\lambda_T) + \ln(\lambda_M)}{\ln\left(\lambda_M \lambda_T^\beta\right)}$ and $\lim\limits_{\Delta t \to 0^+} T_1(\Delta t)^\beta = 0$, we infer that

$$\lim_{\Delta t \to 0^+} \sum_{k \geq 0} \left(I_k(\Delta t) - J_k(\Delta t)\right) = 0.$$

Relation (5.1) is hence still valid here. Furthermore, (5.2) holds for $\mu < 1$. Substitution $u = tT_1(\Delta t)$ leads to

$$\sum_{k \geq 0} J_k(\Delta t) = \frac{T_1(\Delta t)^\mu}{\Delta t \ln(\lambda_T)} \int_{T_1(\Delta t)}^{\infty} u^{-(1+\mu)} \left[N_A(t_0 - u) - N_A(t_0)\right] du.$$

By definition, $0 \leq N_A(t) \leq m'(G)$ for all $t \in \mathbb{R}$. Since we have also assumed that N_A locally satisfies the Hölder condition of order $\nu > \mu$, its Marchaud fractional derivative of order μ is well-defined.

As a consequence,

$$\lim_{\Delta t \to 0^+} \int_{T_1(\Delta t)}^{\infty} u^{-(1+\mu)} \left[N_A(t_0 - u) - N_A(t_0)\right] du = -\frac{\Gamma(1-\mu)}{\mu} \mathcal{D}_+^\mu N_A(t_0).$$

With relation (5.3), we obtain

$$\lim_{\Delta t \to 0^+} \sum_{k \geq 0} J_k(\Delta t) = -\frac{\tau^{\mu-1}}{\ln(\lambda_T)} \frac{\Gamma(1-\mu)}{\mu} \mathcal{D}_+^\mu N_A(t_0).$$

Finally,

$$\begin{aligned} \mathcal{G} N_A(t_0) &= (1 - p_0(0^+)) \frac{\tau^{\mu-1}}{\ln(\lambda_T)} \frac{\ln(\lambda_M) \Gamma(1-\mu)}{\mu} \mathcal{D}_+^\mu N_A(t_0) \\ &= -(1 - p_0(0^+)) \tau^{\mu-1} \Gamma(1-\mu) \mathcal{D}_+^\mu N_A(t_0). \end{aligned}$$

$\square$

We have deliberately let the constant τ appear in (5.4) for reasons of dimensional homogeneity [16]: the relevant derivative is not $\mathcal{D}_+^\mu$, but $\tau^{\mu-1} \mathcal{D}_+^\mu$, in order to be homogeneous to the inverse of a time.

Remark 4. *So as to respect dimensional homogeneity, a constant of time τ' should have been introduced for the traps constants:*

$$\lambda_T(\Delta t) = (\lambda_T)^{\Delta t / \tau'}, \quad \lambda_M(\Delta t) = (\lambda_M)^{\Delta t / \tau'}.$$

However, from remark 2, the transport exponent should have eventually been unchanged.

6. Discussion

6.1. **Characterization of μ.** From distribution $(p_k(\Delta t))_{k\geq 0}$ and characteristic times $(T_k(\Delta t))_{k\geq 0}$ (with $T_0(\Delta t)=0$), we can evaluate moments $\langle T^\alpha\rangle_{\Delta t}$ with $\alpha>0$, defined by

$$\langle T^\alpha\rangle_{\Delta t} = \sum_{k\geq 0} p_k(\Delta t) T_k(\Delta t)^\alpha.$$

In the case of dynamical traps described by (4.5) and (4.7), we obtain:

$$\langle T^\alpha\rangle_{\Delta t} = \begin{cases} (1-p_0(\Delta t))\,\dfrac{1-\lambda_M^{\Delta t}}{1-\left(\lambda_M\lambda_T^\alpha\right)^{\Delta t}} T_1(\Delta t)^\alpha & \text{if } \alpha<\mu, \\ +\infty & \text{if } \alpha\geq\mu. \end{cases}$$

Consequently, if we note $\langle T^\alpha\rangle = \lim\limits_{\Delta t\to 0^+}\langle T^\alpha\rangle_{\Delta t}$, parameter μ appears as a critical point:

$$\langle T^\alpha\rangle = \begin{cases} 0 & \text{if } \alpha<\mu, \\ +\infty & \text{if } \alpha\geq\mu. \end{cases}$$

However, if $m(G)>0$ and $\mu\leq 1$, $\langle T\rangle=\infty$, which does not respect the Kac lemma [17, 19]. Then we should assume $m(G)=0$. This remark is closely akin to the approach of Hilfer [11], where the fractional infinitesimal generator only appears for sets of measure 0.

Remark 5. *If G is a section transverse to the Hamiltonian flow (a Poincaré section for instance) then $m(G)=0$, and since all the trajectories cross G, $\tau_r(x)=0$ for all $x\in G$. Consequently, $G_0(\Delta t)=\emptyset$ and $p_0(\Delta t)=0$, for all $\Delta t>0$. Hence $p_0(0^+)=0$.*

6.2. **Fractional kinetic equation.** The model presented here do not explain the emergence of fractional derivatives in equations such as (4.1). However, the fractional exponent we have obtained is exactly the transport coefficient (4.3). This result is compatible with relation (4.2) which involves the fractional exponents of Zaslavsky.

Indeed, let us assume that $S(\Delta t)$ could be applied to $P(x,t)$ in order to describe "a generalized shift of $P(x,t)$ along t by Δt" [33, p.246]. Then the temporal derivative associated to the temporal evolution of $P(x,t)$ is the infinitesimal generator $\mathcal{G}$. In the case of anomalous diffusion, exponents α and β in (4.1) become completely determined.

- If $\mu>1$ (superdiffusion), then the temporal derivative is classic: $\beta=1$. Superdiffusion is exclusively taken into account by the spatial derivative of order $\alpha=\frac{2}{\mu}$. Equation (4.1) thus become

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^{2/\mu}}{\partial x^{2/\mu}}\left(\mathcal{A}(x)P(x,t)\right).$$

- If $\mu<1$ (subdiffusion), $\beta=\mu$ so $\alpha=2$: the temporal derivative is the only one to be fractional. Consequently, (4.1) turns into

$$\frac{\partial^\mu}{\partial t^\mu}P(x,t) = \frac{\partial^2}{\partial x^2}\left(\mathcal{A}(x)P(x,t)\right).$$

In particular, if our model applies to $P(x,t)$, fractional derivatives in space and time cannot coexist.

7. Conclusion

The model which has been described in this article attempts to explain, from a dynamical view point, the emergence of fractional derivatives in chaotic Hamiltonian systems. It seems simplier than the formalism of Hilfer, in particular because no renormalization appears. Moreover, it strongly relies on fractal properties of the phase space. Our approach is obviously perfectible on several aspects. It does not explain why $T_1(\Delta t)$ should fulfill (5.3), and condition $m(G) = 0$ imposed by the Kac lemma should be clarified. So as to test the validity of the model, other systems should also be considered, in particular strongly chaotic systems, where the distribution of recurrence times is similar to an exponential law [3, 15]. Finally, we believe that there are still enough freedom degrees in our model for allowing us to enhance it in forthcoming studies.

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Institut de Mécanique Céleste et de Calcul des Éphémérides, Observatoire de Paris, 77 avenue Denfert-Rochereau, 75014 Paris, France

E-mail address: pierre.inizan@imcce.fr